

Classification of Arbitrary Multipartite Entangled States under Local Unitary Equivalence

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Abstract

We present a practical method for finding the canonical forms of arbitrary dimensional multipartite entangled states, either pure or mixed. By extending the technique developed in one of our recent works, the canonical forms for arbitrary mixed multipartite entangled states are constructed, which may inherit local symmetries from the $N + 1$ pure states. A systematic scheme to express the local symmetries of the canonical form is proposed, that provides a feasible way to verify the local unitary equivalence for two multipartite entangled states.

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1 Introduction

Entanglement is one of the most important ingredients in quantum information science, gives impetus to the most extraordinary nonclassical applications, such as teleportation and quantum computation, etc [1]. Now it is generally regarded that the entanglement is a key physical resource in realizing many quantum information tasks. Thus the quantitative and qualitative study of entanglement becomes more and more important. Though superficially

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showing up with different features, usually not all these entangled states are functionally independent, they may be intrinsically the same while the entanglement property is concerned. Two entangled states are said to be equivalent in implementing the same quantum information task if they can be obtained with certainty from each other via Local Operation and Classical Communication (LOCC). Theoretically, this LOCC equivalent class is such defined that within this class any two quantum states are inter-convertible by local unitary (LU) operators [2].

The characterization of bipartite entangled states under LU can be well understood by using the singular value (Schmidt) decomposition. However things turn out to be much complicate when the multipartite case is concerned. The characterization of multipartite entanglement can be done by computing the local unitary invariants of the quantum states [3]. Two entangled states are LU equivalent if they have the same LU invariants, the relation between LU equivalence for n -partite pure states and the $(n-1)$ -partite mixed states has also been noticed and is used in constructing the local unitary invariants [4, 5]. The parameters in local invariants grow dramatically as the number of partite increases, and the problem of identifying and interpreting independent invariants becomes very complicated [6]. On the other hand, one can chose certain bases and put the quantum states in some canonical (standard) forms. Along this line, a canonical method was proposed in Ref.[7], though it was only given in a set of constraints on the coefficients of the quantum state. Later, this method was reformulated into a compact form [8]. By introducing the standard form for multipartite states, Kraus proposed a general way to determine the local unitary transformation between two LU equivalent n -qubit states [9], however as the dimension increases, degeneracy emerges between the identical eigenvalues of the one partite reduced density matrix, the verification of LU equivalence becomes unpractical.

Recently, in [10] we have proposed a practical method for finding the canonical form of pure multipartite state by using the high order singular value decompositions and local symmetry properties of the tensor form quantum states. In this work, we generalize this method to the mixed state case where the canonical form for arbitrary mixed multipartite state is constructed. And we also develop a systematic method to represent the local symmetries in the canonical forms, this provides a feasible way to verify the LU equivalence of

two quantum states regardless of the degeneracy conditions.

The structure of the paper goes as follows. In section 2, we give a brief introduction to the basic technique of high order singular value decomposition which is used in the entanglement classification. In section 3, we reformulate the entanglement classification for multipartite pure state under LU in a more neat form, and a practical classification method for arbitrary multipartite mixed states is developed where the canonical form for mixed state is also constructed. After the complete classification of multipartite entangled states (pure or mixed) with their canonical forms, we develop a systematic scheme for verifying the local unitary symmetry between two entanglement classes in section 4. In section 5, practical examples of three and four qubits states are given. Finally, some concluding remarks are presented in section 6.

2 LU equivalence of multipartite quantum state

A general N -partite entangled quantum state in dimension $I_1 \times I_2 \times \cdots \times I_N$ can be formulated in the following form

$$|\Psi\rangle = \sum_{i_1=1, i_2=1, \dots, i_N=1}^{I_1, I_2, \dots, I_N} \psi_{i_1 i_2 \dots i_N} |i_1\rangle |i_2\rangle \dots |i_N\rangle, \quad (1)$$

where $\psi_{i_1 i_2 \dots i_N} \in \mathbb{C}$ are coefficients of the quantum state in representative bases. Two quantum states are said to be LU equivalent if they are inter-convertible by local unitary operators, which can be schematically expressed as:

$$\begin{aligned} |\Psi'\rangle &= \bigotimes_i^N U^{(i)} |\Psi\rangle \\ &= \sum_{\substack{i_1, i_2, \dots, i_N \\ i'_1, i'_2, \dots, i'_N}} \psi_{i_1 i_2 \dots i_N} u_{i'_1 i_1}^{(1)} |i'_1\rangle u_{i'_2 i_2}^{(2)} |i'_2\rangle \cdots u_{i'_N i_N}^{(N)} |i'_N\rangle \\ &= \sum_{i_1, i_2, \dots, i_N} \psi'_{i_1 i_2 \dots i_N} |i_1, i_2, \dots, i_N\rangle. \end{aligned} \quad (2)$$

Here, the coefficients $\psi_{i_1 i_2 \dots i_N}$ can also be treated as the entries of a tensor Ψ , and hence the quantum states can be represented by high dimensional complex tensors. In the tensor form

of Ψ , the unitary operator $U^{(n)}$ acting on the n th partite is defined as

$$(U^{(n)}\Psi)_{i_1 i_2 \dots i_{n-1} i'_n i_{n+1} \dots i_N} \equiv \sum_{i_n} \psi_{i_1 i_2 \dots i_{n-1} i_n i_{n+1} \dots i_N} u_{i'_n i_n}^{(n)}. \quad (3)$$

For bipartite pure state, the tensor Ψ is a matrix $\Psi = [\psi_{i_1 i_2}] \in \mathbb{C}^{I_1 \times I_2}$ (matrices with complex numbers of I_1 rows and I_2 columns) where dimension of the Hilbert space for each partite is I_1 and I_2 separately. The singular value decomposition (SVD) of the bipartite state Ψ of dimensions $I_1 \times I_2$ reads

$$\Lambda = U^{(1)} \cdot \Psi \cdot U^{(2)} = \text{diag}\{\lambda_1, \dots, \lambda_I\}, \quad (4)$$

where $\lambda_i \geq \lambda_j \geq 0, \forall i < j, I = \min\{I_1, I_2\}$. Λ has the following two important properties

1. the singular values $\lambda_i, i \in \{1, \dots, I\}$ of matrix Ψ , are uniquely defined.
2. Λ is a diagonal matrix and uniquely defined (with prescribed order of the singular values).

In this case, the singular values of the quantum state Ψ readily characterize its entanglement properties under LU equivalence. Two bipartite quantum states are LU equivalent if and only if they have the same singular value decompositions.

Here we introduce a generalization of SVD to high dimensional multipartite systems, the high order singular value decomposition (HOSVD), which was presented in [12].

Define the matrix unfolding of the tensor $\Psi \in \mathbb{C}^{I_1 I_2 \dots I_N}$ with n th index as

$$\Psi_{(n)} \in \mathbb{C}^{I_n \times (I_{n+1} I_{n+2} \dots I_N I_1 I_2 \dots I_{n-1})}. \quad (5)$$

Here $\Psi_{(n)}$ is a $I_n \times (I_{n+1} I_{n+2} \dots I_N I_1 I_2 \dots I_{n-1})$ matrix. For example, $2 \times 3 \times 4$ complex tensor Ψ unfolding with the second and third indexes have the following forms

$$\begin{aligned} \Psi_{(2)} &= \begin{pmatrix} \psi_{111} & \psi_{211} & \psi_{112} & \psi_{212} & \psi_{113} & \psi_{213} & \psi_{114} & \psi_{214} \\ \psi_{121} & \psi_{221} & \psi_{122} & \psi_{222} & \psi_{123} & \psi_{223} & \psi_{124} & \psi_{224} \\ \psi_{131} & \psi_{231} & \psi_{132} & \psi_{232} & \psi_{133} & \psi_{233} & \psi_{134} & \psi_{234} \end{pmatrix}, \\ \Psi_{(3)} &= \begin{pmatrix} \psi_{111} & \psi_{121} & \psi_{131} & \psi_{211} & \psi_{221} & \psi_{231} \\ \psi_{112} & \psi_{122} & \psi_{132} & \psi_{212} & \psi_{222} & \psi_{232} \\ \psi_{113} & \psi_{123} & \psi_{133} & \psi_{213} & \psi_{223} & \psi_{233} \\ \psi_{114} & \psi_{124} & \psi_{134} & \psi_{214} & \psi_{224} & \psi_{234} \end{pmatrix}. \end{aligned} \quad (6)$$

For arbitrary N -partite systems there exists a core tensor Ω for each tensor Ψ , that is

$$\Omega = U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)} \Psi . \quad (7)$$

Here Ω is a same order tensor in the Hilbert space $I_1 \times I_2 \times \dots \times I_N$ as Ψ , and any $N-1$ order tensor $\Omega_{i_n=i}$ obtained by fixing the n th index to i , has the following property

$$\langle \Omega_{i_n=i}, \Omega_{i_n=j} \rangle = \delta_{ij} \left(\sigma_i^{(n)} \right)^2 , \quad (8)$$

where $\sigma_i^{(n)}$ is called the n -mode singular value of Ψ and $\sigma_i^{(n)} \geq \sigma_j^{(n)} \geq 0, \forall i < j$. The singular value $\sigma_i^{(n)}$ symbolizes the Frobenius-norm $\sigma_i^{(n)} = \|\Omega_{i_n=i}\| \equiv \sqrt{\langle \Omega_{i_n=i}, \Omega_{i_n=i} \rangle}$, where the inner product $\langle \mathcal{A}, \mathcal{B} \rangle \stackrel{\text{def}}{=} \sum_{i_1} \sum_{i_2} \dots \sum_{i_N} b_{i_1 i_2 \dots i_N} a_{i_1 i_2 \dots i_N}^*$ (see [12] for details of the technique).

In the following we show how we get the core tensor by the LU transformation $U^{(i)}, i \in \{1, \dots, N\}$ in Eq.(7). Consider a quantum state Ψ with dimension of $I_1 \times I_2 \times \dots \times I_N$, another quantum state Ω with the same dimension as Ψ is LU equivalent to Ψ if

$$\Omega = U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)} \Psi , \quad (9)$$

where $U^{(i)}, i \in \{1, \dots, N\}$ are unitary matrices. In the matrix unfolding form, Eq.(9) can be rewritten as

$$\Omega_{(n)} = U^{(n)} \cdot \Psi_{(n)} \cdot (U^{(n+1, \dots, n-1)})^T . \quad (10)$$

Here $U^{(n+1, \dots, n-1)} \equiv U^{(n+1)} \otimes U^{(n+2)} \otimes \dots \otimes U^{(N)} \otimes U^{(1)} \otimes \dots \otimes U^{(n-1)}$, and $\Omega_{(n)}, \Psi_{(n)}$ have the same dimensions: I_n rows and $(I_{n+1} \times I_{n+2} \times \dots \times I_N \times I_1 \times \dots \times I_{n-1})$ columns. Now consider the particular case where $U^{(n)}$ is obtained from the singular value decomposition of matrix $\Psi_{(n)}$, i.e.,

$$U^{(n)} \cdot \Psi_{(n)} \cdot V^{(n)} = \text{diag}\{\sigma_1^{(n)}, \sigma_2^{(n)}, \dots, \sigma_{I_n}^{(n)}\} = \Sigma_{(n)} , \quad (11)$$

where $U^{(n)}$ and $V^{(n)}$ are unitary matrix, and $\sigma_i^{(n)} \geq \sigma_j^{(n)} \geq 0, \forall i < j$. Then Eq.(10) can be written as

$$\Omega_{(n)} = \Sigma_{(n)} \cdot V^{(n)\dagger} \cdot (U^{(n+1, \dots, n-1)})^T . \quad (12)$$

It is clear that $\Omega_{(n)}$ has orthogonal rows

$$\langle \Omega_{i_n=j}, \Omega_{i_n=k} \rangle = \delta_{jk} \left(\sigma_j^{(n)} \right)^2. \quad (13)$$

And Eq.(13) always holds if $U^{(n+1, \dots, n-1)}$ is unitary matrix. In the similar way we can construct other local unitary matrices $U^{(i)}, i \in \{1, 2, \dots, N\}$. Eventually, all the $U^{(i)}$ can be obtained, the core tensors Ω of Ψ can then be constructed via Eq.(9).

From the construction of the core tensor, two of the important properties of HOSVD compared to its bipartite counterpart can be concluded:

1. The n -mode singular values $\sigma_i^{(n)}, i \in \{1, \dots, I_n\}, n \in \{1, \dots, N\}$, of Ψ are uniquely defined.
2. If $\forall n \in \{1, \dots, N\}$ the n -mode singular values $\sigma_i^{(n)}$ are all distinct, then $\Omega'_{(n)} = \Theta_{(n)} \Omega_{(n)}$ is also a HOSVD of Ψ where $\Theta_{(n)} = \text{diag}\{e^{i\theta_1^{(n)}}, \dots, e^{i\theta_{I_n}^{(n)}}\}$. Otherwise, let $\sigma_1^{(n)} > \sigma_2^{(n)} > \dots > \sigma_{k_n}^{(n)} \geq 0$ denote the distinct n -mode singular values of $\Omega_{(n)}$ with respective positive multiplicities $\mu_1^{(n)}, \mu_2^{(n)}, \dots, \mu_{k_n}^{(n)}$ where $\sum_{i=1}^{k_n} \mu_{i_n}^{(n)} = I_n$. In this case if $\Omega_{(n)}$ is a HOSVD of Ψ then

$$\Omega'_{(n)} = \left[\bigoplus_{i=1}^{k_n} u_{k_n}^{(n)} \right] \Omega_{(n)} \equiv S^{(n)} \Omega_{(n)}, \quad (14)$$

is also a HOSVD of Ψ . Here $u_{k_n}^{(n)} \in \mathbb{C}^{\mu_{k_n}^{(n)} \times \mu_{k_n}^{(n)}}$ are arbitrary unitary matrix and the diagonal blocks of $S^{(n)}$ conformal to those n -mode singular values with multiplicity.

From the second property it is clear that, unlike the bipartite case, the core tensor Ω (HOSVD) of Ψ is not uniquely defined.

3 Classification under local unitary equivalence

In this section we propose entanglement classification scheme by decomposing the LU equivalence of the quantum states into two correlated problems: the HOSVD and local unitary symmetries. First, we give a brief introduction to the entanglement classification

of arbitrary dimensional multipartite pure states which was first proposed in [10], then we generalize the method to the mixed states, by which the canonical forms for entanglement classes of mixed states under the LU equivalence can be constructed neatly.

3.1 The equivalence under LU for multipartite pure states

Due to the nonuniqueness of the core tensors, Ω can not be identified as the entanglement classes of the quantum states. The philosophy of our scheme in [10] is that if we impose this nonuniqueness as a local symmetry within the core tensors themselves, then we can get the unique canonical forms. That is if we regard the core tensors Ω and Ω' which are related by local unitary operators as the same entanglement class then the HOSVD can be seen as the entanglement classification of the multipartite state Ψ .

Suppose that the core tensor Ω have k_n distinct n -mode singular values $\sigma_i^{(n)}, i \in \{1, 2, \dots, k_n\}$, each with multiplicity of $\mu_{i_n}^{(n)}$ where $\sum_{i_n=1}^{k_n} \mu_{i_n}^{(n)} = I_n$. Here we regard these multiplicities as the degeneracies of the singular values which corresponds to the case of nongeneric states of [9]. From Eq.(14) we can infer that the local unitary symmetry which relates two core tensors takes the following form

$$S = \bigotimes_{n=1}^N \left[\bigoplus_{j_n=1}^{k_n} u_{j_n}^{(n)} \right]. \quad (15)$$

The core tensors Ω' and Ω related by this symmetry now can be written as

$$\Omega' = S \Omega. \quad (16)$$

Two different core tensors related by S belong to the same entanglement class, we can call the core tensors Ω of Ψ associated with corresponding local symmetry S the canonical form of Ψ .

In order to see how this symmetry act on the core tensors we introduce the technique of vectorization of the matrix. With each matrix $A = [a_{ij}] \in \mathbb{C}^{I_1 \times I_2}$, we can associated it with a vector \vec{A} defined by

$$\vec{A} \equiv [a_{11}, \dots, a_{I_1 1}, a_{12}, \dots, a_{I_1 2}, \dots, a_{1 I_2}, \dots, a_{I_1 I_2}]^T. \quad (17)$$

Two tensors Ψ and Ψ' of $I_1 \times I_2 \times \cdots \times I_N$ which are related by local operators $U^{(n)}, n \in \{1, 2, \dots, N\}$, can be expressed in the matrix unfolding form with the n th index

$$\Psi'_{(n)} = U^{(n)} \cdot \Psi_{(n)} \cdot (U^{(n+1)} \otimes U^{(n+2)} \otimes \cdots \otimes U^{(N)} \otimes U^{(1)} \otimes \cdots \otimes U^{(n-1)})^T, \quad (18)$$

With the convention of Eq.(17), the matrix equation Eq.(18), can be written as (see [13])

$$U^{(n+1)} \otimes U^{(n+2)} \otimes \cdots \otimes U^{(N)} \otimes U^{(1)} \otimes \cdots \otimes U^{(n-1)} \otimes U^{(n)} \vec{\Psi}_{(n)} = \text{vec } \vec{\Psi}'_{(n)}. \quad (19)$$

This can be seen as a unitary transformation of a $I_1 \times I_2 \times \cdots \times I_N$ vector $\vec{\Psi}_{(n)}$ to $\vec{\Psi}'_{(n)}$.

In case of $n = N$ we have the simple form of Eq.(19)

$$U^{(1)} \otimes \cdots \otimes U^{(N)} \vec{\Psi}_{(N)} = \text{vec } \vec{\Psi}'_{(N)}. \quad (20)$$

In this form, the symmetry between the core tensors can be represented as

$$\vec{\Omega}'_{(N)} = S \vec{\Omega}_{(N)} \equiv \bigotimes_{n=1}^N \left[\bigoplus_{j_n=1}^{k_n} u_{j_n}^{(n)} \right] \vec{\Omega}_{(N)}, \quad (21)$$

where $u_{j_n}^{(n)}$ is unitary matrix of $\mu_{j_n}^{(n)} \times \mu_{j_n}^{(n)}$ and $\sum_{j_n=1}^{k_n} \mu_{j_n}^{(n)} = I_n$. In the blocks diagonalized form, Eq.(21) is

$$\begin{pmatrix} u_1^{(1)} \otimes \cdots \otimes u_1^{(N)} & 0 & \cdots & 0 \\ 0 & u_1^{(1)} \otimes \cdots \otimes u_2^{(N)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{k_1}^{(1)} \otimes \cdots \otimes u_{k_N}^{(N)} \end{pmatrix} \cdot \vec{\Omega}_{(N)} = \vec{\Omega}'_{(N)}. \quad (22)$$

We can set $u_{j_n}^{(n)} = e^{i\theta_{j_n}^{(n)}}$ if the multiplicity $\mu_{j_n}^{(1)} = 1$. In all, we have the following theorem which has been state in [10]

Theorem 1 *The core tensor Ω associated with the local symmetry group S is the canonical form of the multipartite pure state and is the entanglement class under LU equivalence.*

From a more general point of view, any quantum state Ψ associated with a local unitary group $U = \bigotimes_i U^{(i)}$ can also be seen as the unique entanglement class. In this unified scenario,

we can compare the symmetry U and S corresponding to Ψ and Ω separately

$$\begin{aligned} S &= \bigotimes_{n=1}^N \left[\bigoplus_{j_n=1}^{k_n} u_{j_n}^{(n)} \right] \\ &= \bigoplus_{j_1, j_2, \dots, j_N=1}^{k_1, k_2, \dots, k_N} u_{j_1}^{(1)} \otimes u_{j_2}^{(2)} \otimes \dots \otimes u_{j_N}^{(N)}, \end{aligned} \quad (23)$$

$$U = \bigotimes_{n=1}^N U^{(n)}, \quad (24)$$

The total matrix dimension of these two groups of S and U are

$$\begin{aligned} \text{Dim}[S] &= \sum_{j_1, j_2, \dots, j_N} \prod_{n=1}^N \mu_{j_n}^{(n)} = \prod_{n=1}^N \sum_{j_n} \mu_{j_n}^{(n)} \\ &= \prod_{n=1}^N I_n = \text{Dim}[U]. \end{aligned} \quad (25)$$

Through this comparison we can see that, a arbitrary quantum state Ψ under LU transformation U forms an entanglement class $U\Psi$ which is also a orbit under the transformation U . After the entanglement classification we arrive at the canonical form of the quantum state, the admissible transformation S which keeps the canonical form invariant is regarded as the symmetry of the canonical form. In Eq.(23), S can be seen as a conjugate class of U under a particular local unitary transformation. In the special case that all the singular values are distinct for each partite, the symmetry becomes

$$S = \bigotimes_{n=1}^N \left[\bigoplus_{j_n=1}^{I_n} e^{i\theta_{j_n}^{(n)}} \right], \quad (26)$$

which is just the conjugate class of $\bigotimes_{i=1}^N U^{(i)}$ under $\bigotimes_{i=1}^N U^{(i)}$.

From the quantum state point of view, tensor Ω now is decomposed into several invariant subtensors (denoted it with ω) of the Hilbert space of $I_1 \times I_2 \times \dots \times I_N$ under the transformation S . The dimensions of these subtensors conformal to the direct summed subgroups of S in Eqs.(22, 23). For example Eq.(22) can be written as

$$\begin{pmatrix} u_1^{(1)} \otimes \dots \otimes u_1^{(N)} & 0 & \dots & 0 \\ 0 & u_1^{(1)} \otimes \dots \otimes u_2^{(N)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{k_1}^{(1)} \otimes \dots \otimes u_{k_N}^{(N)} \end{pmatrix} \cdot \begin{pmatrix} \vec{\omega}_{r_1} \\ \vec{\omega}_{r_2} \\ \vdots \\ \vec{\omega}_{r_m} \end{pmatrix} = \begin{pmatrix} \vec{\omega}'_{r_1} \\ \vec{\omega}'_{r_2} \\ \vdots \\ \vec{\omega}'_{r_m} \end{pmatrix}, \quad (27)$$

where $\vec{\omega}_{r_i}$ and $\vec{\omega}'_{r_i}$ are the segments of the column vectors $\vec{\Omega}_{(N)}$ and $\vec{\Omega}'_{(N)}$ with the dimension conformal to the diagonal blocks $u_{i_1}^{(1)} \otimes \cdots \otimes u_{i_n}^{(N)}$. $\vec{\omega}$ are just the vector forms of the subtensors ω .

3.2 The equivalence under LU for multipartite mixed states

The classification of the entanglement for mixed states is generally believed to be more complicated than pure states in many cases. However in the case of LU equivalence, it has been noticed that n -partite pure state is related to its $(n - 1)$ -partite mixed state [5]. Here we generalize our entanglement classification method developed for multipartite pure states to the case of arbitrary dimensional N -partite mixed states.

Consider a mixed N -partite quantum state ρ which is generally expressed as

$$\rho = \sum_i p_i^2 |\psi_i\rangle\langle\psi_i|, \quad (28)$$

where $\sum_i p_i^2 = 1$, $p_i \in \mathbb{R}^+$, $|\psi_i\rangle$ are N partite pure states. We add an additional 0th partite to the original N -partite mixed state ρ and formulate a $N + 1$ pure quantum state in the following form

$$\Psi_0 = \sum_i p_i |i\rangle |\psi_i\rangle, \quad (29)$$

where $|i\rangle$ is the 0th partite. For this quantum state, we have the following fact

$$\begin{aligned} \text{Tr}_0 [|\Psi_0\rangle\langle\Psi_0|] &= \sum_{n,i,j} p_i p_j \langle n|i\rangle |\psi_i\rangle\langle\psi_j| \langle j|n\rangle \\ &= \sum_{n,i,j} p_i p_j \langle j|n\rangle \langle n|i\rangle |\psi_i\rangle\langle\psi_j| \\ &= \sum_i p_i^2 |\psi_i\rangle\langle\psi_i| = \rho. \end{aligned} \quad (30)$$

And further we have, if $\Psi'_0 = U^{(0)} \otimes E^{(1)} \otimes \cdots \otimes E^{(N+1)} \Psi_0 \equiv U^{(0)} \Psi_0$, where E is unit matrix,

then

$$\begin{aligned}
& \text{Tr}_0 [|\Psi'_0\rangle\langle\Psi'_0|] = \text{Tr}_0 [U^{(0)}|\Psi_0\rangle\langle\Psi_0|U^{(0)\dagger}] \\
&= \sum_{n,i,j} p_i p_j \langle n|U^{(0)}|i\rangle |\psi_i\rangle\langle\psi_j| \langle j|U^{(0)\dagger}|n\rangle \\
&= \sum_{n,i,j} p_i p_j \langle j|U^{(0)\dagger}|n\rangle \langle n|U^{(0)}|i\rangle |\psi_i\rangle\langle\psi_j| \\
&= \rho = \text{Tr}_0 [|\Psi_0\rangle\langle\Psi_0|] .
\end{aligned} \tag{31}$$

From the above two facts we can state that the following relation

$$\rho = \sum_{i=1}^r p_i^2 |\psi_i\rangle \xrightleftharpoons[\text{Tr}_0]{+0} \Psi = \sum_{i=1}^r p_i |i\rangle |\psi_i\rangle \tag{32}$$

forms a bijection between ρ and Ψ_0 . Define this bijection as a map between Ψ_0 and ρ , we have the following proposition

Proposition 2 *An arbitrary dimensional mixed N -partite state ρ' is LU equivalent to ρ , i.e.,*

$$\rho' = U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)} \rho U^{(1)\dagger} \otimes U^{(2)\dagger} \otimes \dots \otimes U^{(N)\dagger} \tag{33}$$

if and only if its pure state counterpart Ψ'_0 is LU equivalent to Ψ_0 , that is

$$\Psi'_0 = U^{(0)} \otimes U^{(1)} \otimes \dots \otimes U^{(N)} \Psi_0 . \tag{34}$$

Proof: The relation between N partite ρ and $(N+1)$ -partite Ψ_0 is

$$\rho = \sum_{i=1}^r p_i^2 |\psi_i\rangle \xrightleftharpoons[\text{Tr}_0]{+0} \Psi_0 = \sum_{i=1}^r p_i |i\rangle |\psi_i\rangle , \tag{35}$$

$$\rho' = \sum_{j=1}^{r'} q_j^2 |\psi'_j\rangle \xrightleftharpoons[\text{Tr}_0]{+0} \Psi'_0 = \sum_{j=1}^{r'} q_j |j\rangle |\psi'_j\rangle . \tag{36}$$

First, if

$$\begin{aligned}
\rho' &= U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)} \rho U^{(1)\dagger} \otimes U^{(2)\dagger} \otimes \dots \otimes U^{(N)\dagger} \\
&= \sum_{i=1}^r p_i^2 U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)} |\psi_i\rangle\langle\psi_i| U^{(1)\dagger} \otimes U^{(2)\dagger} \otimes \dots \otimes U^{(N)\dagger} \\
&= \sum_{i=1}^r p_i^2 |\psi'_i\rangle\langle\psi'_i| ,
\end{aligned} \tag{37}$$

where $|\psi'_i\rangle = U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)} |\psi_i\rangle$, then Ψ'_0 correspond to ρ' is

$$\begin{aligned}
\Psi'_0 &= \sum_{j=1}^r p_j |j\rangle |\psi'_j\rangle = \sum_{j=1}^r p_j |j\rangle U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)} |\psi_j\rangle \\
&= E^{(0)} \otimes U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)} \sum_{j=1}^r p_j |j\rangle |\psi_j\rangle \\
&= E^{(0)} \otimes U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)} \Psi_0.
\end{aligned} \tag{38}$$

That is Ψ'_0 is LU equivalent to Ψ_0 .

Second if $\Psi'_0 = U^{(0)} \otimes \dots \otimes U^{(N)} \Psi_0$, then

$$\begin{aligned}
\rho' &= \text{Tr}_0 [|\Psi'_0\rangle\langle\Psi'_0|] \\
&= \text{Tr}_0 \left[\sum_{i=1, j=1}^r p_i p_j U^{(0)} |i\rangle U^{(1)} \otimes \dots \otimes U^{(N)} |\psi_i\rangle \langle\psi_j| U^{(1)\dagger} \otimes \dots \otimes U^{(N)\dagger} \langle j| U^{(0)\dagger} \right] \\
&= \text{Tr}_0 \left[\sum_{i=1, j=1}^r p_i p_j |i\rangle U^{(1)} \otimes \dots \otimes U^{(N)} |\psi_i\rangle \langle\psi_j| U^{(1)\dagger} \otimes \dots \otimes U^{(N)\dagger} \langle j| \right] \\
&= U^{(1)} \otimes \dots \otimes U^{(N)} \left[\sum_{i=1}^r p_i^2 |\psi_i\rangle \langle\psi_i| \right] U^{(1)\dagger} \otimes \dots \otimes U^{(N)\dagger} \\
&= U^{(1)} \otimes \dots \otimes U^{(N)} \rho U^{(1)\dagger} \otimes \dots \otimes U^{(N)\dagger}.
\end{aligned} \tag{39}$$

Here, we have used the fact of Eq.(31). That is ρ' is LU equivalent to ρ . QED.

We can now conclude that: if ρ' is LU equivalent to ρ then their corresponding pure states Ψ'_0 and Ψ_0 can be related by LU operators; if Ψ' is LU equivalent to Ψ_0 then their reduced matrices ρ' and ρ are LU equivalent. We may construct the core tensor Ω_0 from Ψ_0 , then we trace out the 0th partite from the core tensor Ω_0 and obtain the canonical form for ρ , that is $\Upsilon = \text{Tr}_0 [|\Omega_0\rangle\langle\Omega_0|]$

Theorem 3 *The canonical form Υ is of the entanglement class of mixed state ρ up to a inherit local symmetry from Ω_0 .*

This method provide a sample way to construct the canonical form for the mixed N -partite state ρ : first construct the $N+1$ partite pure state Ψ_0 from ρ ; then compute the core tensor Ω_0 of Ψ_0 ; finally we arrive the canonical form by tracing out the 0th partite $\Upsilon = \text{Tr}_0 |\Omega_0\rangle\langle\Omega_0|$.

4 The local symmetries of the canonical form

We have constructed the canonical forms for both pure and mixed multipartite states. In all, the construction of the canonical forms will result in a general form of Eq.(22) whether the state is pure or not. With this detailed form of the symmetries, in this section we develop a practical scheme to verify the LU equivalence of two quantum states which have the same singular values and same degeneracies for each partite.

4.1 A general form of the local unitary symmetry

We start from a general case, that is we have k_n distinct n -mode singular values $\sigma_{i_n}^{(n)}, i_n \in \{1, 2, \dots, k_n\}$, each with multiplicities of $\mu_{i_n}^{(n)}$ where $\sum_{i_n=1}^{k_n} \mu_{i_n}^{(n)} = I_n$. Define the n -mode singular value vector $\vec{\sigma}^{(n)}$ for the matrix unfolding form of $\Omega_{(n)}$

$$\begin{aligned} \vec{\sigma}^{(n)} \equiv & \left\{ \underbrace{\sigma_1^{(n)}, \sigma_1^{(n)}, \dots, \sigma_1^{(n)}}_{\mu_1^{(n)}}, \underbrace{\sigma_2^{(n)}, \sigma_2^{(n)}, \dots, \sigma_2^{(n)}}_{\mu_2^{(n)}}, \right. \\ & \vdots \\ & \left. \underbrace{\sigma_{k_n}^{(n)}, \sigma_{k_n}^{(n)}, \dots, \sigma_{k_n}^{(n)}}_{\mu_{k_n}^{(n)}} \right\}^T, \end{aligned} \quad (40)$$

where $\sigma_i^{(n)} > \sigma_j^{(n)} \geq 0, \forall i < j$. The local symmetry corresponding to this partite is

$$S^{(n)} \equiv \begin{pmatrix} u_1^{(n)} & & \\ & \ddots & \\ & & u_{k_n}^{(n)} \end{pmatrix}, \quad (41)$$

where $u_{i_n}^{(n)}, i_n \in \{1, \dots, k_n\}$ are unitary with the dimension of $\mu_{i_n}^{(n)} \times \mu_{i_n}^{(n)}$.

The total local unitary symmetry $S = \bigotimes_i S^{(i)}$ of the core tensors Ω is

$$\begin{pmatrix} u_1^{(1)} & & \\ & \ddots & \\ & & u_{k_1}^{(1)} \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} u_1^{(N)} & & \\ & \ddots & \\ & & u_{k_N}^{(N)} \end{pmatrix} \cdot \vec{\Omega}_{(N)} = \vec{\Omega}'_{(N)}, \quad (42)$$

which is just Eq.(22). Similarly the total singular values can be expressed as a matrix

$$\Sigma \equiv \{\vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}, \dots, \vec{\sigma}^{(N)}\}. \quad (43)$$

We can call it the “singular value matrix” of the core tensor which is uniquely defined according to the properties of HOSVD. Quantum states with different singular value matrices are apparently LU inequivalent. Then the core tensors which have the same singular value matrix belong to the same entanglement class if and only if they satisfy Eq.(27). In Eq.(27), the verification of the LU equivalence of two core tensors turns to finding solutions for the following equation group with varying r

$$u_{i_1}^{(1)} \otimes u_{i_2}^{(2)} \otimes \cdots \otimes u_{i_N}^{(N)} \vec{\omega}_r = \vec{\omega}_r . \quad (44)$$

This can be seen as a fine grained LU classification problem of the subtensor ω_r . Thus we can pick the sub tensor ω_r out of the tensor Ω and do the HOSVD to it recursively.

As the fine grained process goes, the recursion will end at two cases: 1, the singular values are all distinct for all the partite; 2, the singular values are all the same for all the partite.

For the first case, if $\forall i_n \in \{1, \dots, k_n\}$ and $\forall n \in \{1, \dots, N\}$, the singular value multiplicity $\mu_{i_n}^{(n)} = 1$, then $k_n = I_n$ and

$$U^{(n)} = \begin{pmatrix} \exp(i\theta_1^{(n)}) & 0 & \cdots & 0 \\ 0 & \exp(i\theta_2^{(n)}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp(i\theta_{I_n}^{(n)}) \end{pmatrix} , \quad (45)$$

Eq.(27) turns to

$$\exp[i(\theta_{i_1}^{(1)} + \theta_{i_2}^{(2)} + \cdots + \theta_{i_N}^{(N)})] \omega_{i_1 i_2 \dots i_N} = \omega'_{i_1 i_2 \dots i_N} . \quad (46)$$

Here we write $\omega_{i_1 i_2 \dots i_N}$ instead of $\vec{\omega}_{i_1 i_2 \dots i_N}$ because $\omega_{i_1 i_2 \dots i_N}$ now is a complex number in this case. A log operation on Eq.(46) will make

$$\theta_{1i_1} + \theta_{2i_2} + \cdots + \theta_{Ni_N} = -i \log \left[\frac{\omega'_{i_1 i_2 \dots i_N}}{\omega_{i_1 i_2 \dots i_N}} \right] . \quad (47)$$

These are $I_1 \times I_2 \times \cdots \times I_N$ linear equations for $I_1 + I_2 + \cdots + I_N$ phase variables, it can be verified immediately whether they have consistent solutions. The quantum states is LU equivalent if and only if there is at least one solution to this linear equation group.

4.2 Completely degenerate state for all the partite

In the completely degenerate state, the reduced density matrix for each partite is proportional to unit matrix. Consider a arbitrary N -partite pure state with dimension of $I_1 \times I_2 \times \cdots \times I_N$. The complete degenerate state is that $\forall n \in \{1, 2, \cdots, N\}$

$$\rho_n = \text{Tr}_{-n} [|\Omega\rangle\langle\Omega|] = \frac{1}{I_n} E. \quad (48)$$

The core tensor has the following form

$$\begin{aligned} \Omega &= \sum_{i_n} |i_n\rangle \sum_{\neg i_n} \omega_{i_1 i_2 \cdots i_{n-1} i_n i_{n+1} \cdots i_N} |i_1 i_2 \cdots i_{n-1} i_{n+1} \cdots i_N\rangle \\ &= \sum_{i_n} |i_n\rangle |\omega_{\neg n}^{(i_n)}\rangle, \quad n \in \{1, \cdots, N\}, \end{aligned} \quad (49)$$

where $\langle \omega_{i_n'}^{(i_n)} | \omega_{\neg n}^{(i_n)} \rangle = \frac{1}{I_n} \delta_{i_n i_n'}$. The local symmetry S takes the following form

$$S \cdot \vec{\Omega}_{(N)} = \bigotimes_n U^{(n)} \cdot \vec{\Omega}_{(N)} = \vec{\Omega}'_{(N)}. \quad (50)$$

An arbitrary unitary matrix is unitarily equivalent to a diagonal matrix, that is

$$U^{(n)} = X^{(n)\dagger} \cdot \Phi^{(n)} \cdot X^{(n)}, \quad (51)$$

where $X^{(n)}$ is unitary matrix and $\Phi^{(n)} = \text{diag}\{e^{i\phi_1^{(n)}}, \cdots, e^{i\phi_{I_n}^{(n)}}\}$ is the conjugate class of $U^{(n)}$.

Eq.(50) now turns to

$$\bigotimes_n \Phi^{(n)} \cdot \bigotimes_n X^{(n)} \cdot \vec{\Omega} = \bigotimes_n X^{(n)} \cdot \vec{\Omega}'. \quad (52)$$

The Eq.(52) corresponds to $I_1 \times I_2 \times \cdots \times I_N$ homogeneous equations, which in the detailed form one of the of these equations of Eq.(52) looks like

$$\sum_{i_1 \cdots i_N} x_{j_1 i_1}^{(1)} x_{j_2 i_2}^{(2)} \cdots x_{j_N i_N}^{(N)} \cdot (e^{i(\phi_{j_1}^{(1)} + \phi_{j_2}^{(2)} + \cdots + \phi_{j_N}^{(N)})} \omega_{i_1 i_2 \cdots i_N} - \omega'_{i_1 i_2 \cdots i_N}) = 0. \quad (53)$$

Here we represent x_{ij} as the elements of matrix X . This is a typical equation group of $I_1 \times I_2 \times \cdots \times I_N$ equations for $I_1^2 + I_2^2 + \cdots + I_N^2$ complex parameters $x_{i_n i_n'}^{(n)}$ (note we first solve the parameters $x_{ij}^{(n)}$ then impose the unitary condition on the matrix $X^{(n)}$).

For this kind of nonlinear equations there exist simple tool called “linearization” or “relinearization” [14, 15]. The key algorithm rely on the fact that for $N > 2$ multipartite quantum states, when the dimensional or number of partite increases, the number of equations grows much more quickly than the number of the parameters. Generally Eq.(53) would turn out to be a over defined system of equations which mean that there are more equations than unknow parameters.

The linearization technique goes as follows. Regard each monomial of the matrix elements as a individual variable

$$\nu_{i_1 i_2 \dots i_N, i'_1 i'_2 \dots i'_N} = x_{i_1 i'_1}^{(1)} x_{i_2 i'_2}^{(2)} \dots x_{i_N i'_N}^{(N)}, \quad (54)$$

then there will be $(I_1 \times I_2 \times \dots \times I_N)^2$ such variables ν . Eq.(53) now can be written as

$$\left[\sum_{i_1 \dots i_N} \nu_{j_1 j_2 \dots j_N, i_1 i_2 \dots i_N} (e^{i(\phi_{j_1}^{(1)} + \phi_{j_2}^{(2)} + \dots + \phi_{j_N}^{(N)})} \omega_{i_1 i_2 \dots i_N} - \omega'_{i_1 i_2 \dots i_N}) \right] = 0. \quad (55)$$

For the sake of simplicity we use the convention that \mathbf{i} represents the value of the bit string $(i_1 i_2 \dots i_N)$, i.e., $\mathbf{i} = 1 = (11 \dots 1)$ and $\mathbf{i} = 2 = (11 \dots 2)$, etc. Define $\omega_{\mathbf{j}\mathbf{i}} \equiv e^{i(\phi_{j_1}^{(1)} + \phi_{j_2}^{(2)} + \dots + \phi_{j_N}^{(N)})} \omega_{\mathbf{i}} - \omega'_{\mathbf{i}}$ where $\mathbf{j} = (j_1 j_2 \dots j_N)$. Eq.(55) can be reformulated as

$$\omega_{\mathbf{j}\mathbf{i}} \cdot \nu_{\mathbf{j}\mathbf{i}} = 0, \quad (56)$$

where the dot means the summation over \mathbf{i} . Taking a $2 \times 2 \times 2$ system as an example, we have

$$\begin{aligned} &\omega_{\mathbf{j}1} \nu_{\mathbf{j}1} + \omega_{\mathbf{j}2} \nu_{\mathbf{j}2} + \omega_{\mathbf{j}3} \nu_{\mathbf{j}3} + \omega_{\mathbf{j}4} \nu_{\mathbf{j}4} + \\ &\omega_{\mathbf{j}5} \nu_{\mathbf{j}5} + \omega_{\mathbf{j}6} \nu_{\mathbf{j}6} + \omega_{\mathbf{j}7} \nu_{\mathbf{j}7} + \omega_{\mathbf{j}8} \nu_{\mathbf{j}8} = 0. \end{aligned} \quad (57)$$

There are 8 such equations for \mathbf{j} runs from 1 to 8. The solution can be expressed as

$$\begin{pmatrix} \nu_{\mathbf{j}1} \\ \nu_{\mathbf{j}2} \\ \nu_{\mathbf{j}3} \\ \vdots \\ \nu_{\mathbf{j}8} \end{pmatrix} = c_{\mathbf{j}2} \begin{pmatrix} -\frac{\omega_{\mathbf{j}2}}{\omega_{\mathbf{j}1}} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_{\mathbf{j}3} \begin{pmatrix} -\frac{\omega_{\mathbf{j}3}}{\omega_{\mathbf{j}1}} \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + c_{\mathbf{j}8} \begin{pmatrix} -\frac{\omega_{\mathbf{j}8}}{\omega_{\mathbf{j}1}} \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad (58)$$

where $c_{\mathbf{j}i}$ are new parameters. Clearly, Eq.(58) is a under defined equation group for parameters $\nu_{\mathbf{j}i}$. However there are additional equations between the products of $\nu_{\mathbf{j}i}$ s, i.e.,

$$\begin{aligned} & \nu_{i_1 \dots i_n \dots i_N, j_1 \dots j_n \dots j_N} \nu_{i'_1 \dots i'_n \dots i'_N, j'_1 \dots j'_n \dots j'_N} \\ &= \nu_{i_1 \dots i'_n \dots i_N, j_1 \dots j'_n \dots j_N} \nu_{i'_1 \dots i_n \dots i'_N, j'_1 \dots j_n \dots j'_N} . \end{aligned} \quad (59)$$

This relation is inherit from Eq.(54) and the following fact

$$\begin{aligned} & x_{i_1 j_1}^{(1)} \dots x_{i_n j_n}^{(n)} \dots x_{i_N j_N}^{(N)} \cdot x_{i'_1 j'_1}^{(1)} \dots x_{i'_n j'_n}^{(n)} \dots x_{i'_N j'_N}^{(N)} \\ &= x_{i_1 j_1}^{(1)} \dots x_{i'_n j'_n}^{(n)} \dots x_{i_N j_N}^{(N)} \cdot x_{i'_1 j'_1}^{(1)} \dots x_{i_n j_n}^{(n)} \dots x_{i'_N j'_N}^{(N)} . \end{aligned} \quad (60)$$

For example in $2 \times 2 \times 2$ system we have $\nu_{111,111} \nu_{111,122} = \nu_{111,121} \nu_{111,112}$ or simply $\nu_{11} \nu_{14} = \nu_{13} \nu_{12}$. This imposes additional equation between parameters $\nu_{\mathbf{j}i}$, and can also be viewed as an equation in the (smaller number of) parameters $c_{\mathbf{j}i}$ expressing them. The new system of equations derived from all the possible relations of the type of Eq.(59). In solving the equations on $c_{\mathbf{j}i}$ we can use the linearization method recursively.

Here we give a explicit formula for how many constrains of Eq.(59) there will be. As there are $(I_1 \times I_2 \times \dots \times I_N)^2$ matrix elements, we can get $(I_1 \times I_2 \times \dots \times I_N)^2$ new parameters $\nu_{\mathbf{j}i}$. If we multiply m times of $\nu_{\mathbf{j}i}$, that is

$$\underbrace{\nu_{\mathbf{j}i} \dots \nu_{\mathbf{j}'i'}}_m , \quad (61)$$

we will have

$$C_{(I_1 \times I_2 \times \dots \times I_N)^2 + m - 1}^m \quad (62)$$

different productions. On the contrary, according to the productions of $x_{ij}^{(n)}$, the actual number of different productions is only

$$\prod_{i=1}^N C_{I_i^2 + m - 1}^m , \quad (63)$$

which is much less than the number of equations (here $C_n^l = \frac{n!}{l!(n-l)!}$). The number of Eq.(62) is greater than that of Eq.(63) when $m > 1$. For the case of $2 \times 2 \times 2$ and $m = 2$ we have

$$C_{(2 \times 2 \times 2)^2 + 2 - 1}^2 = 2080 , \quad (C_{2^2 + 2 - 1}^2)^3 = 1000 , \quad (64)$$

which means that we have 2080 different $\nu_{\mathbf{j}i} \nu_{\mathbf{j}'i'}$ s, but only 1000 are independent. A considerably large amount of constraint equations like Eq.(59) are obtained.

5 Examples of the canonical form for three and four qubits state

Here we give two simple examples of how we can get the canonical forms of the arbitrary quantum state, and how we can verify whether two quantum states in the canonical forms can be related by symmetry S . As the entanglement classification of the mixed states can be reduced to specific pure states case, here we only give examples of pure states.

We can randomly generate a $2 \times 2 \times 2$ pure state Ψ with its matrix unfolding

$$\Psi_{(1)} = \begin{pmatrix} 0.0260603 & 1.05491 & -3.69051 & 0.437711 \\ 1.25266 & 1.07259 & 3.2378 & 1.5625 \end{pmatrix}. \quad (65)$$

From the algorithm of Eq.(12), the singular matrix is

$$\begin{pmatrix} \sigma_1^{(1)} & \sigma_1^{(2)} & \sigma_1^{(3)} \\ \sigma_2^{(1)} & \sigma_2^{(2)} & \sigma_2^{(3)} \end{pmatrix} = \begin{pmatrix} 5.03906 & 5.31586 & 5.17055 \\ 2.27534 & 1.5202 & 1.95825 \end{pmatrix}. \quad (66)$$

The core tensors is then

$$\Omega_{(1)} = \begin{pmatrix} -5.01792 & 0.2815 & -0.354882 & -0.0862168 \\ 0.19519 & 1.72088 & -1.17941 & -0.886923 \end{pmatrix}. \quad (67)$$

We give another example of four qubits state. Two $2 \times 2 \times 2 \times 2$ quantum states

$$\begin{aligned} \Psi_{(1)} &= \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \\ \Psi'_{(1)} &= \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \end{aligned} \quad (68)$$

which are already the core tensors. The singular value matrices for them are the same

$$\begin{pmatrix} \sigma_1^{(1)} & \sigma_1^{(2)} & \sigma_1^{(3)} & \sigma_1^{(4)} \\ \sigma_2^{(1)} & \sigma_2^{(2)} & \sigma_2^{(3)} & \sigma_2^{(4)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{4}{5} & \frac{4}{5} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{5} & \frac{1}{5} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \sigma_1'^{(1)} & \sigma_1'^{(2)} & \sigma_1'^{(3)} & \sigma_1'^{(4)} \\ \sigma_2'^{(1)} & \sigma_2'^{(2)} & \sigma_2'^{(3)} & \sigma_2'^{(4)} \end{pmatrix}. \quad (69)$$

In the vector forms of the matrices unfolding of $\Psi_{(1)}$ and $\Psi'_{(1)}$, the symmetry S takes the following form

$$S = \begin{pmatrix} e^{i\theta_1^{(2)}+i\theta_1^{(3)}} U^{(4)} \otimes U^{(1)} & 0 & 0 & 0 \\ 0 & e^{i\theta_1^{(2)}+i\theta_2^{(3)}} U^{(4)} \otimes U^{(1)} & 0 & 0 \\ 0 & 0 & e^{i\theta_2^{(2)}+i\theta_1^{(3)}} U^{(4)} \otimes U^{(1)} & 0 \\ 0 & 0 & 0 & e^{i\theta_2^{(2)}+i\theta_2^{(3)}} U^{(4)} \otimes U^{(1)} \end{pmatrix}.$$

The core tensors are then divided into four segments correspondingly

$$\vec{\omega}_1 = \frac{1}{\sqrt{10}} \text{diag}\{1, 0, 0, 1\} \quad \vec{\omega}' = \frac{1}{\sqrt{10}} \text{diag}\{1, 0, 0, 1\} , \quad (70)$$

$$\vec{\omega}_1 = \frac{1}{\sqrt{10}} \text{diag}\{0, 0, 0, 0\} \quad \vec{\omega}'_2 = \frac{1}{\sqrt{10}} \text{diag}\{0, 0, 0, 0\} , \quad (71)$$

$$\vec{\omega}_3 = \frac{1}{\sqrt{10}} \text{diag}\{0, 0, 0, 0\} \quad \vec{\omega}'_3 = \frac{1}{\sqrt{10}} \text{diag}\{0, 0, 0, 0\} , \quad (72)$$

$$\vec{\omega}_4 = \frac{1}{\sqrt{10}} \text{diag}\{2, 0, 0, 2\} \quad \vec{\omega}'_4 = \frac{1}{\sqrt{10}} \text{diag}\{2, 0, 0, -2\} , \quad (73)$$

Take all the above equations into Eq.(27) we have the following effective equations

$$e^{i\theta_1^{(2)}+i\theta_1^{(3)}} U^{(4)} \otimes U^{(1)} \vec{\omega}_1 = \vec{\omega}'_1 , \quad (74)$$

$$e^{i\theta_2^{(2)}+i\theta_2^{(3)}} U^{(4)} \otimes U^{(1)} \vec{\omega}_4 = \vec{\omega}'_4 . \quad (75)$$

The components of $\vec{\omega}_1$, $\vec{\omega}'_1$ and $\vec{\omega}_4$, $\vec{\omega}'_4$ in Eqs.(70,73) bring a contradiction to Eqs.(74,75). Now it is clear that there is no solutions for $U^{(4)}$ and $U^{(1)}$, and thus the four qubits states $\Psi_{(1)}$ and $\Psi'_{(1)}$ are LU inequivalent.

6 Conclusions

In summary, by using the tensor decomposition method we generalize the entanglement classification under LU equivalence to arbitrary dimensional multipartite mixed states. The classification actually reduces to the construction of the canonical forms of the corresponding $N + 1$ -partite pure states. With the analysis of the local symmetry in the canonical form, the core tensor can be decomposed into a series of subtensors which are transformed independently under the local symmetry. Base on this recognition of the entanglement structure, a practical scheme is also developed for the verification of local unitary equivalence of two multipartite entangled states.

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